

Fractal Potential Flows as an Exact Model for Fully Developed Turbulence

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Abstract

Fully Developed Turbulence (FDT) occurs at the infinite extreme of the Reynolds spectrum. It is a theoretical phenomenon which can only be approximated experimentally or computationally, and thus its precise properties are only hypothetical, though widely accepted. It is considered to be a chaotic yet steady flow field, with self-similar fractalline features. A number of approximate models exist, often exploiting this self-similarity. We hereby present the exact mathematical model of Fractal Potential Flows, and link it philosophically to the phenomenon of FDT, building on its experimental characteristics. The model hinges on the recursive iteration of a fluid dynamical transfer operator. We show the existence of its unique attractor in an appropriate function space - called the invariant flow - which will serve as our model for the FDT flow field. Its sink singularities are shown to form an IFS fractal, resolving Mandelbrot's Conjecture. Meanwhile we present an isometric isomorphism between flows and probability measures, hinting at a wealth of future research.*

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1 Introduction

We begin our discussion in Section 2.1 with an extensive overview of Fully Developed Turbulence (FDT), in an attempt to identify the definitive characteristics of this hypothetical phenomenon at the infinite end of the Reynolds spectrum, which can only be approximated and observed via experiments or simulation. We then proceed to give an account of the beautiful idea of Potential Flows and their superposition, heretofore seemingly unrelated to FDT, yet intriguingly sharing its inviscid and steady characteristics. We proceed to detailing the influential direction of non-deterministic experimental Fluid Dynamics, called Chaotic Advection, which we shall reformulate in the language of Optimal Transport Theory. Since the well-known IFS invariant measure is the fixed point of the iterative evolution under the measure transfer operator, it will serve as our inspiration for showing the unique existence of an invariant flow. Since the transfer operator distributes singularities over an IFS fractal, this adds impetus to pursuing a transfer model, considering Mandelbrot’s Conjecture. In our remarks on this overview, we make a decisive step to discard Navier-Stokes evolution on mathematical grounds, observing the non-smooth nature of intermittency, and turn our attention to discrete inertial transfer evolution on piecewise steady flows.

The fusion of the above theories will come naturally and effortlessly in the form of Fractal Potential Flows, introduced in Section 3. The random intermittent interaction of a finite system of eddies, is directly modelled via the recursive iteration of a weighted transfer operator in Section 3.2. The contractive action of each eddy sink on the flow field, is modelled with a corresponding pushforward transfer map, induced by a contraction map of an IFS. Since the collective action of the eddies is probabilistically weighted, it can be interpreted as an “expected value” expression, while physically speaking as the weighted effect each eddy has on the flow field. The iteration of the transfer operator, is reasoned to correspond to the non-smooth discrete energy bursts of intermittency.

The evolution equation towards FDT having been reasoned to be the discrete transfer recursion, our goal consequently becomes to show that it converges to a unique attractor flow - the FDT flow field itself - in some appropriate function space. In finding the correct space, eddies remain our guiding inspiration, considering that a system of eddies becomes a superposed eddy when zoomed out, possessing a complex character at complex infinity as discussed in Section 3.3. In their local universe, the eddies interact to produce an increasingly fractalline flow field, approaching the desired invariant flow shown in Section 3.5. The required iteration number to reach FDT is infinite, while the total time can be potentially finite, if the time-spacing is for instance in geometric regression, as discussed in Section 3.6.

Lastly, the sinks of the invariant flow field are shown to form an IFS fractal in Section 3.7.1, implying the ultimate conclusion that the geometrical study of such sets, is of fundamental relevance to analyzing an FDT flow field.

2 Preliminary Concepts

2.1 Fully Developed Turbulence

Turbulence is a challenging phenomenon to model, primarily because it is difficult to define rigorously. One must isolate from experimental observation its definitive characteristics, before attempting to model it. However, even its main characteristics are difficult to identify, and one may never feel secure that some crucial characteristic may have been excluded from the assumptions of a model, or that some assumption has been made which is not in agreement with reality. Furthermore, one must carefully differentiate between observed and intuitive characteristics as described by other researchers, as we hereby will. One must also restrict the problem of modelling sufficiently, in order for a specific solution to exist, which is why we shall focus on Fully Developed Turbulence (FDT). Although other approximate models may exist, one only needs to consider the phenomenon itself as defined from experiments, when attempting to build a new model. Currently FDT has no known exact model, which is a major issue considering its widespread occurrence in Nature and technology.

A rich literature exists on the topic of FDT, and we recommend the overview of Ecke [6] for an introduction. Our overview assumes that the reader is familiar with elementary Fluid Dynamics.

2.1.1 Definition

Turbulence has been the interest of classical physics for centuries. Its name comes from the Italian word “turbolenza” given by Leonardo da Vinci, one of the first to study and depict the phenomenon. In terms of the Reynolds number, flows above an Re of 5000 or higher are considered to be turbulent.

The phenomenon of FDT is an extremal case of turbulence, based on experimental observations at high Reynolds numbers, typically above 10,000. The words “fully developed” in FDT refer to its main theoretical characteristics: an infinite Reynolds number and a steady unchanging yet chaotic velocity field, independent of viscosity. This final unchanging nature of the field is meant to characterize an equilibrium state or invariance with respect to its evolution equation, whatever it may be. Furthermore, the velocity field, its set of singularities, as well as its evolution is hypothesized to be self-similar or fractalline in some sense.

The precise ideal conditions for FDT to occur are currently unknown, and its exact causes, evolution, and characteristics are hypothetical. With some contemplation, one may realize that a mathematical theory is the only possible resolution. It is currently believed that the Navier-Stokes Equations can potentially describe turbulence at high Reynolds numbers.

2.1.2 Relevance

The modelling of turbulence - whether approximate or exact - is of tremendous importance, considering its various and widespread applications.

The study of atmospheric and oceanic turbulence occurs in the context of Geophysical Fluid Dynamics and Climate Theory - primarily in the form of 2D turbulence. Quantum turbulence in quantum fluids, such as superfluids, occurs at close to zero viscosity. Magnetohydrodynamic (MHD) turbulence is relevant to astro- and geophysics, ranging from the study of planetary magnetic fields to large-scale industrial facilities, and plasma flow in stars and fusion reactors. In biology, plankton distribution is influenced by turbulence, and is important since plankton comprise the bottom of the oceanic food chain.

2.1.3 Characteristics, Causes, and Evolution

Turbulence literature describes a set of observed and intuitive characteristics, and one must carefully distinguish between them. The causes and evolution towards Fully Developed Turbulence are also hypothetical, yet reasonable and experimentally inspired. Observed characteristics include high and erratic velocity fluctuation over a large range of coupled spatial and temporal scales. Consequently, a turbulent flow field exhibits structure at many length scales, often described as self-similar or scale-independent in some sense.

The onset and evolution of turbulence can be fuelled by various circumstances, such as the interaction of eddies in the flow, by injected kinetic energy into the flow such as via stirring, or even by the shape or surface of the container. Examples include flow from a faucet, flow past rocks or a curved wall, or the rotation of the Earth. A periodicity in the circumstances often plays a role in inducing intermittency. Mathematically speaking, development can occur due to periodic changes in the external forcing, or variations in the boundary conditions. In intermittency, transitions are observed to occur abruptly to successively more complex states. The evolution of turbulence is suspected to be purely inertial, meaning due to the actual motion of the fluid, even if fuelled externally. Flows where inertial effects are small, tend to remain laminar.

The induced eddies and their intermittent inertial interaction are considered to be the intrinsic elements of turbulent dynamics. Since the injected energy at the top scale is periodic, the eddies in their evolutionary hierarchy receive periodic bursts of energy as well, causing their interaction to be intermittent. In between each energy burst, a brief quiescent period occurs, where the fluid flows along its natural course i.e. its steady streamlines.

A hypothetical picture of the eddy hierarchy is the idea of vortex stretching and the Richardson energy cascade. The eddies on the free surface of the fluid, will have axes roughly perpendicular to the surface. A thinning of the eddies will occur due to the conservation of fluid elements. This results in the breakdown of larger structures into hierarchically smaller ones, causing a cascade of energy. This is a purely inertial and inviscid process, which con-

tinues until the local structures are small enough that viscosity causes their kinetic energy to dissipate as heat.

Conjecture 2.1 (Mandelbrot [11]) *At infinite Reynolds numbers (FDT), the energy dissipation of the fluid concentrates in a set of non-integer Hausdorff dimension.*

The conjecture has been resolved by V. Scheffer in 1980, though the proof is non-constructive. If one wishes to construct an FDT flow field, this conjecture may serve as an inspiration.

The Richardson cascade is primarily a 2D phenomenological hypothesis, taking place on the surface of the fluid. Turbulent fluids typically occupy 3D space, but motion in the vertical direction can become less relevant, and we are left with a quasi-2D hydrodynamics. The degree to which the third dimension is suppressed, primarily depends on the ratio of the vertical to the lateral scales, and the effect of external forces causing dominant lateral velocities over vertical ones.

2.1.4 Existing Models

Previous efforts at modelling turbulence, and FDT in particular, have been approximate or statistical in nature, mostly in the form of scaling laws derived from the equations of motion. This direction was inspired by Kolmogorov’s influential model based on the Richardson cascade, culminating in the current Beta Model developed by Novikov, Stewart, Mandelbrot, Kraichan, Frisch, Sulem and Nelkin et al. For a detailed overview, we recommend Jou [8]. There is also the computational direction of simulating large Reynolds number flows on high-performance computers, originally suggested by John von Neumann.

The theory of Chaotic Advection is an experimental and mathematical direction, meant to investigate the intermittent interaction of eddies directly. Its language is Potential Flow theory and Dynamical Systems. We will dedicate an entire section to its review.

Detailed understanding and prediction from first principles still elude turbulence theory, which mostly bases itself on the Navier-Stokes Equations. Currently, it is considered to be highly unlikely that an exact rigorous mathematical model should be possible for either turbulence and its evolution, or its end result, Fully Developed Turbulence itself. If such a model exists, it is conjectured to be geometrical. Nevertheless, an exact model must only be in agreement with the observed phenomenon itself, and not with other derived models.

2.2 Potential Flows

For laminar flows outside the boundary layer, potential flows are considered to provide a sufficient model. Yet the turbulent boundary layer itself is often modelled approximately with a superposition of vortices (Vortex Dynamics). These flows are assumed to be steady (time-independent), ideal (zero viscosity / inviscid, uniform density, and incompressible), and irrotational (Lebesgue almost everywhere, denoted as a.e.).

By these conditions, conservation of mass for a velocity field $v = (v_1, v_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is

$$0 = \operatorname{div} v = \partial_1 v_1 + \partial_2 v_2 \quad \text{a.e.}$$

and the equations of motion become

$$\frac{1}{2} \nabla |v|^2 - v \times \operatorname{curl} v = -\frac{1}{\varrho} \nabla p$$

where ϱ is the density and p is pressure, further reducing to

$$p(z) = \left(p(z_0) + \frac{\varrho}{2} |v(z_0)|^2 \right) - \frac{\varrho}{2} |v(z)|^2$$

under the irrotationality requirement

$$0 = \operatorname{curl} v = \partial_2 v_1 - \partial_1 v_2 \quad \text{a.e.}$$

Supposing that $\psi, \phi \in C_{ac}^2(\mathbb{R}^2, \mathbb{R})$ are harmonic conjugates, meaning they satisfy the Cauchy-Riemann Equations

$$\Delta \psi = \Delta \phi = 0 \quad \text{a.e.} \quad \text{and} \quad \partial_1 \phi = \partial_2 \psi, \quad \partial_2 \phi = -\partial_1 \psi \quad \text{a.e.}$$

then $v := -\nabla \phi = i \nabla \psi$ satisfies mass conservation and irrotationality a.e. The pressure field can be calculated from v as above. We call such a.e. harmonic ψ for which a conjugate exists, the stream function of a potential flow, and ϕ the potential function. We note that the potential function corresponding to a stream function is only unique up to a gradient. Nevertheless we denote it as $\tilde{\psi} := \phi$, and denote equivalence in the gradient as

$$\phi_1 \equiv \phi_2 \Leftrightarrow \nabla \phi_1 = \nabla \phi_2 \quad \text{a.e.}$$

A harmonic function is known to admit a conjugate if its domain is simply connected. Furthermore $\tilde{\tilde{\psi}} = -\psi$.

By their relation to the velocity field, the curves of constant ψ represent the streamlines of the flow, and the curves of constant ϕ the equipotential lines. Having two pairs of harmonic conjugates ψ_1, ϕ_1 and ψ_2, ϕ_2 their linear combinations $a\psi_1 + b\psi_2$ and $a\phi_1 + b\phi_2$ for any $a, b \in \mathbb{R}$ are also harmonic conjugates. So the principle of superposition holds, as long as the boundary conditions are also correspondingly combined. Since the equations of motion have a unique solution for a set of boundary conditions, and they translate to Laplace's equation (the solution of which for appropriate boundary conditions also exists and is unique up to a gradient), we may conclude that potential flows fully characterize the set of all flows which are steady, ideal, and irrotational a.e.

Elementary potential flows, which are often superposed to create more complex ones, include sinks, sources and the (circular) vortex. The stream and potential functions of a source ($q > 0$) or a sink ($q < 0$) with strength $q \in \mathbb{R}$ are defined at $z \in \mathbb{C}$ as

$$\psi(z) = -\frac{q}{2\pi} \operatorname{Arg} z \bmod |q| \quad \text{and} \quad \phi(z) = -\frac{q}{2\pi} \ln |z|$$

while for the orthogonal circular vortex

$$\psi(z) = \frac{\Gamma}{2\pi} \ln |z| \quad \text{and} \quad \phi(z) = -\frac{\Gamma}{2\pi} \text{Arg } z \bmod |\Gamma|$$

where Γ is the circulation around any closed path containing the vortex (strength of the vortex).

Superposing a sink and a circular vortex with a general center $p \in \mathbb{C}$ results in a logarithmic vortex (eddy) as follows

$$\begin{aligned} \psi(z) &= -\frac{q}{2\pi} \text{Arg}(z - p) + \frac{\Gamma}{2\pi} \ln |z - p| \bmod |q| \\ \phi(z) &= -\frac{q}{2\pi} \ln |z - p| - \frac{\Gamma}{2\pi} \text{Arg}(z - p) \bmod |\Gamma| \end{aligned}$$

Since ψ is differentiable in the \mathbb{R}^2 sense almost everywhere (except at p and along the branch $\psi(z) = q$) we have that the corresponding velocity field (extended continuously to the entire plane) is the following

$$v(z) = \frac{q + \Gamma i}{2\pi} \frac{1}{|z - p|} \frac{z - p}{|z - p|} \quad (z \in \mathbb{C})$$

Here we consider the gradient vectors to be on the complex plane and differentiation in the bivariate sense, meaning $\nabla \psi = \partial_1 \psi + i \partial_2 \psi$. In general, we will sloppily identify $z \mapsto \psi(z)$, $z \in \mathbb{C}$ with $(x, y) \mapsto \psi(x + yi)$, $(x, y) \in \mathbb{R}^2$ when it is more convenient to do so.

2.3 Chaotic Advection

2.3.1 Aref's Blinking Vortex-Sink System

Advection is the idea of a fluid transporting light matter (typically tracer particles) on its free surface, so that the velocity of the particles is given by the velocity field of the flow. If ψ is the stream function of the velocity field, then this corresponds to the Hamiltonian equations $v_1 = -\partial_2 \psi$, $v_2 = \partial_1 \psi$. Hence the configuration space of an advected particle is the phase space of this Hamiltonian system. Often such a system exhibits chaos, or sensitive dependence on initial conditions. The notion that laminar flows can produce chaotic particle trajectories, is considered today to be a cornerstone of Fluid Dynamics. Hassan Aref is the originator of this idea, and his influence has propagated far.

The flows mostly studied in this context, are piecewise steady flows. Meaning that on the evolution timeline of the flow, the velocity field is steady for disjoint consecutive time intervals, and a non-smooth abrupt transition (jump) to a new velocity field occurs at each partitioning moment in time. The jumps act along the streamlines of two circular (or logarithmic) vortices, which take alternating (non-random) turns in perturbing the flow field. This is called the Blinking Vortex(-Sink) System, and has been examined extensively by a number of theorists and experimentalists. It is meant to directly simulate the intermittent

interaction of (circular or) logarithmic vortices, which is considered to be the elementary driving mechanism of the evolution towards Fully Developed Turbulence.

Aref in his highly influential paper [1] originally defined the setup with two circular vortices, which has been modified in [2] to two vortex-sinks (eddies), and was further examined by Károlyi and Tél [10] as well as a number of other enlightened researchers.

In particular, Wiggins [20] discusses the method and implications of modelling this blinking as the alternating (non-random) infinite iterative composition of two maps (called linked twist maps, the study of which goes back to Devaney [5]) in order to show the existence of a chaotic invariant set via the Conley-Moser conditions. These ideas are further examined with Ottino in [21], and bear an inspirational resemblance to our own, to be presented here. For an overview of Chaotic Advection (or Mixing), we recommend the book by Ottino [13].

2.3.2 Ott’s Chaos Game

Ott et al. [22] has made a significant improvement on Aref’s model. The regular iterative alternation between two sinks has been modified to a random iteration over a finite family of mappings that transform the flow at the intermittent time partitions, in between steady flow periods. So at step $l \in \mathbb{N}$ (time partition t_l) the state of the flow σ_l (usually the position of a tracer particle on the fluid surface) is transformed as $\sigma_{l+1} = T_l(\sigma_l)$ where $T_l \in \{T_1, \dots, T_n\}$ is chosen at random, and where the precise form of T_l can vary and is often left implicit. This random discrete evolution is reminiscent of the Chaos Game of Barnsley [3]. Though as we will see, this formulation can be closely aligned with experiments, it is difficult however to treat it mathematically. Ott et al. further assume that t_l are spaced evenly and their stream functions are time-dependent, as typical in the theory of Chaotic Advection. However as we will see, both of these are unnecessary constraints, in our explicit and more natural formulation of this chaos game, in Section 3.2. An overview of the work of Ott and Sommerer et al. in collaboration with Tél has been presented in [16], building upon [22, 15, 14].

2.3.3 Sommerer’s Experiment

Chaotic Advection is a de facto experimental theory, and its predictions have been verified by both computational and real-world experiments. Sommerer’s apparatus [15, 16] carried out at the JHU Applied Physics Laboratory, consists of a tank of water with a set of sources at the bottom, which transmit pulsing injections from an external pulsatile flow modulator that randomly regulates which source receives the next injection at equal time intervals, circulated by an external pump. Each pulse corresponds to an iterative step $\sigma_{l+1} = T_l(\sigma_l)$ where the fixed set of $T_l \in \{T_1, \dots, T_n\}$ represents the actions of each source on the particles. These fluorescent particles on the fluid surface are then traced over their chaotic paths. In a sense, this apparatus is an inversion of the Aref setup, since it utilizes sources instead of sinks. Nevertheless, its relevance lies in attempting to simulate the intermittent evolution of turbulence, in the spirit of Ott’s chaos game recursion.

2.4 IFS Fractals

The attractors of Iterated Function Systems - IFS Fractals - were pioneered by Hutchinson [7], further discussed by Barnsley and Demko [4], and may be the most elementary fractals possible. They are the attractors of a finite set of affine linear contraction mappings on the plane - the “function system” - which when combined and iterated to infinity, converges to an attracting limit set, the IFS fractal itself. Our formulation follows [17].

2.4.1 Definition and Existence

Definition 2.1 *Let a 2-dimensional affine contractive mapping (briefly contraction or contraction map) $T : \mathbb{C} \rightarrow \mathbb{C}$ be defined for all $z \in \mathbb{C}$ as $T(z) := p + \varphi(z - p)$ where $p \in \mathbb{C}$ is the fixed point of T , and $\varphi = \lambda e^{i\vartheta} \in \mathbb{C}$ is the factor of T , with $\lambda \in (0, 1)$ the contraction factor of T , and $\vartheta \in (-\pi, \pi]$ the rotation angle of T .*

Note that an equivalent definition may be given using unitary rotation matrices $R \in \mathbb{R}^{2 \times 2}$, $R^T R = I$ corresponding to $e^{i\vartheta}$. Then the contraction maps take the form

$$T(z) = p + \lambda R(z - p) \quad (z \in \mathbb{R}^2, p \in \mathbb{R}^2, \lambda \in (0, 1))$$

This version shall be useful for certain proofs in later sections.

Definition 2.2 *Let a 2-dimensional affine contractive n -map iterated function system (briefly IFS or n -map IFS, $n \in \mathbb{N}$) be defined as a finite set of contractions, and denoted as $\mathcal{T} := \{T_1, \dots, T_n\}$. We will denote the set of indices as $\mathcal{N} := \{1, \dots, n\}$, the set of fixed points as $\mathcal{P} := \{p_1, \dots, p_n\}$, and the set of factors as $\Phi := \{\varphi_1, \dots, \varphi_n\}$.*

Definition 2.3 *Let $\mathcal{T} = \{T_1, \dots, T_n\}$, $n \in \mathbb{N}$ be an IFS. We define the Hutchinson operator H belonging to \mathcal{T} as*

$$H(S) := \bigcup_{k=1}^n T_k(S) \quad \text{where} \quad T_k(S) := \{T_k(z) : z \in S\} \quad \text{for any} \quad S \subset \mathbb{C}$$

and call $H(S)$ the Hutchinson of the set S .

Theorem 2.1 (Hutchinson [7]) *For any IFS with Hutchinson operator H , there exists a unique compact set $F \subset \mathbb{C}$ such that $H(F) = F$. Furthermore, for any compact $S_0 \subset \mathbb{C}$, the recursive iteration $S_{n+1} := H(S_n)$ converges to F in the Hausdorff metric.*

Proof The proof follows from the Banach Fixed Point Theorem, once we show that H is contractive in the Hausdorff metric over compact sets, which form a complete space. \square

Definition 2.4 *Let the set F in the above theorem be called a fractal generated by an IFS with Hutchinson operator H (briefly IFS fractal). Denote $\langle T_1, \dots, T_n \rangle = \langle \mathcal{T} \rangle := F$.*

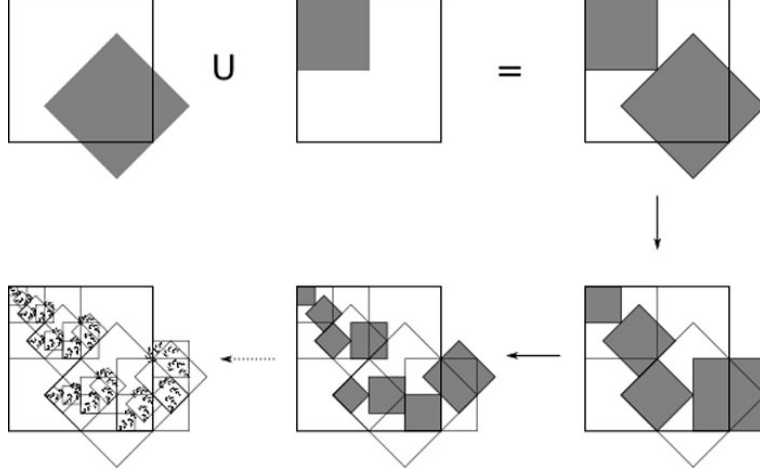


Figure 1: Generation of a fractal by iterating a square (created by Scott Draves).

2.4.2 The Address Set

The address set arises from the iteration of the Hutchinson operator, and it is a way to label each fractal point. Since we can start the iteration towards F with any compact set, we will choose the primary fixed point for simplicity, which is any point in \mathcal{P} of our preference.

Definition 2.5 Let $\mathcal{N}^L := \mathcal{N} \times \dots \times \mathcal{N}$ be the index set to the L -th Cartesian power, and call this $L \in \mathbb{N}$ the iteration level. Then define the address set as $\mathcal{A} := \{0\} \cup \bigcup_{L=1}^{\infty} \mathcal{N}^L$. For any $a \in \mathcal{A}$ denote its k -th coordinate as $a(k)$, $k \in \mathbb{N}$. Let its dimension or length be denoted as $|a| \in \mathbb{N}$ so that $a \in \mathcal{N}^{|a|}$ and let $|0| := 0$. Define the map with address $a \in \mathcal{A}$ acting on any $z \in \mathbb{C}$ as the function composition $T_a(z) := T_{a(1)} \circ \dots \circ T_{a(|a|)}(z)$. Let the identity map be $T_0 := Id$. For a set of weights $\{w_1, \dots, w_n\} \subset (0, 1)$ define $w_a := w_{a(1)} \cdot \dots \cdot w_{a(|a|)}$, for a set of factors $\{\varphi_1, \dots, \varphi_n\}$ define $\varphi_a := \varphi_{a(1)} \cdot \dots \cdot \varphi_{a(|a|)}$, and for a set of angles $\{\vartheta_1, \dots, \vartheta_n\}$ define $\vartheta_a := \vartheta_{a(1)} + \dots + \vartheta_{a(|a|)}$.

Theorem 2.2 For any primary fixed point $p \in \mathcal{P}$ we have

$$F = \langle T_1, \dots, T_n \rangle = \lim_{L \rightarrow \infty} H^L(\{p\}) = \text{Cl}\{T_a(p) : a \in \mathcal{A}\} = \text{Cl}\{T_a(p_k) : a \in \mathcal{A}, p_k \in \mathcal{P}\}$$

We call this the address generation of F .

Proof The proof follows from Theorem 2.1 with either of the initial sets $\{p\}$ or \mathcal{P} . \square

Definition 2.6 Inspired by the above theorem, we will call $F_L := \{T_a(p) : a \in \mathcal{A}, |a| = L\}$ for some $L \in \mathbb{N}$ and primary fixed point $p \in \mathcal{P}$ the L -th level iterate towards $(T_1, \dots, T_n) := \{T_a(p) : a \in \mathcal{A}\}$ the countably infinite IFS fractal.

2.5 Optimal Transport over Measures

Transportation Theory, often referred to as the theory of Optimal Transport, is the study of the optimal transportation and allocation of resources. Given two finite sets of producers and consumers equal in number, with each producer supplying a specific consumer according to a bivariate cost function for transportation, the goal is to find the minimal arrangement between producers and consumers, called the optimal transport plan or map.

The discrete problem can be generalized to probability measures - the Monge-Kantorovich formulation, named after its two most influential creators. Like many optimization problems, this minimization problem has a dual maximization equivalent. When the cost function over a compact metric space (X, d_X) is the metric itself, then the dual becomes the first Wasserstein distance (also called Hutchinson metric) between the elements of the set of Borel probability measures $\nu_{1,2} \in \mathcal{M}_X$ over X , defined as

$$d(\nu_1, \nu_2) = \sup \left\{ \int_X f \, d(\nu_1 - \nu_2) : f \in \mathcal{L}_1(X) \right\}$$

where $\mathcal{L}_1(X)$ are the Lipschitz-1 functions over X . That this is indeed the dual to the minimization problem, was shown by Kantorovich and Rubinstein [9]. Furthermore (\mathcal{M}_X, d) is known to be a complete metric space.

Generally the transport plan is a pushforward map (transfer operator) of the form $T^*\nu = \nu \circ T^{-1}$ ($\nu \in \mathcal{M}_X$). If T^* is contractive in d , then by the Banach Fixed Point Theorem it has a unique invariant measure $\bar{\nu} = T^*\bar{\nu}$. Taking a finite weighted average of such pushforward operators, the resulting transfer operator

$$T = w_1 T_1^* + \dots + w_n T_n^* \quad \text{where} \quad w_k \in (0, 1), \quad w_1 + \dots + w_n = 1$$

will also be contractive in d over \mathcal{M}_X , and possesses a fixed point. Thus the following theorems hold for an IFS $\{T_1, \dots, T_n\}$ in the form $T_k(z) = p_k + \lambda_k R_k(z - p_k)$, $z \in \mathbb{R}^2$ supposing that $F \subset X \subset \mathbb{R}^2$.

Theorem 2.3 (Hutchinson [7]) *For any IFS $\{T_1, \dots, T_n\}$ and weights $\{w_1, \dots, w_n\}$, the transfer operator $T = w_1 T_1^* + \dots + w_n T_n^*$ is contractive on (\mathcal{M}_X, d) and attains a unique invariant probability measure $\bar{\nu} = T\bar{\nu}$ over the compact set $X \subset \mathbb{R}^2$, with support $\langle T_1, \dots, T_n \rangle$. We call this the invariant measure with respect to the IFS $\{T_1, \dots, T_n\}$. For any $\nu_0 \in \mathcal{M}_X$ the recursion $\nu_{L+1} = w_1 T_1^* \nu_L + \dots + w_n T_n^* \nu_L \in \mathcal{M}_X$, $L \in \mathbb{N}$ converges to the invariant measure.*

We remark that with the primary fixed point p and the notations of Section 2.4.2, the measures

$$\nu_0 = \delta_p, \quad \nu_L = \sum_{|a|=L} w_a T_a^* \delta_p \in \mathcal{M}_X, \quad L \in \mathbb{N}$$

satisfy the above recursion, and so $\bar{\nu}(S) = \lim(\nu_L)$ as discussed in [17]. We also observe that the support of ν_L is concentrated on the L -th iterate $F_L := H^L(\{p\}) = \{T_a(p) : a \in \mathcal{A}, |a| = L\}$ approaching the compact IFS fractal $F = \langle T_1, \dots, T_n \rangle$.

We further remark that defining

$$\mathcal{D}_X := \{\varrho \in C_{ac}(\mathbb{R}^2, \mathbb{R}) : \varrho \geq 0, \text{ Supp } \varrho \subset X, \int_X \varrho = 1\}$$

$$\mathcal{P}_X := \left\{ \nu \in \mathcal{M}_X : \exists \varrho \in \mathcal{D}_X \forall S \subset X \text{ Borel set} : \nu(S) = \int_S \varrho \right\}$$

and keeping the above metric d , then $\text{Conv}\{\delta_p : p \in \mathbb{R}^2\} \subset \mathcal{P}_X \subset \mathcal{M}_X$ and $\bar{\nu} \in \mathcal{P}_X$, so it remains true that there exists a unique invariant measure in (\mathcal{P}_X, d) , and it is the same as the earlier. Furthermore $T : \mathcal{P}_X \rightarrow \mathcal{P}_X$ since $\text{Supp}(T\nu) = H(\text{Supp } \nu) \subset X$ if $\nu \in \mathcal{P}_X$.

The study of transfer operators - also called Ruelle(-Perron-Frobenius-Markov) operators - is a rich field, and their largest eigenvalue is typically one, while their eigenfunctions are usually fractalline or self-similar in some sense. This has profound implications for classical mechanics, such as the increase of entropy or the irreversibility of time. An eigenvalue of one, corresponds to a state of equilibrium.

For a more complete discussion of Optimal Transport, we refer the reader to Villani [18] and McCann [12]. For an overview of IFS, we recommend articles [19, 7, 4].

2.6 Remarks

We hereby wish to make some remarks on the previous sections, which will shape our approach to be presented in Section 3.

First of all, it seems reasonable to raise the doubt whether Fully Developed Turbulence in full three dimensions even exists. Certainly, flows can approach above 10,000 Reynolds numbers, but it is doubtful whether the other defining characteristics of FDT, including the equilibrium state of an unchanging velocity field at $Re = \infty$ that is independent of viscosity, can evolve in a fully three-dimensional flow field. Furthermore, we recall that the Richardson cascade that inspired the notion of self-similarity, is primarily a 2D picture on the free surface of the fluid. It seems that 3D space has one too many degrees of freedom, while a limiting 2D free surface forces the fluid to self-organize via self-similarity. Thus we restrict our attention to the 2D free surface in our effort to model FDT, in the spirit of Chaotic Advection.

Secondly, we point out the mathematically unreasonable effort of various other approaches, which attempt to model FDT via the Navier-Stokes Equations. The discussed energy bursts fuelling the erratic characteristic of intermittency, are clear signs of the non-smooth nature of the evolution towards FDT. The Navier-Stokes Equations however, require temporal differentiability of the velocity field, which does not hold by the observed phenomenon of intermittent evolution, summarized above. Therefore one has no choice mathematically, but to discard the Navier-Stokes Equations, in hope of a more fitting evolution equation, in line with the experimental picture of intermittent inertial evolution.

Thirdly, we point out that prior to the advent of Chaotic Advection, potential flows had been considered to be useless for the study of turbulence. Yet ironically, potential flows share with FDT the characteristics of independence with respect to viscosity and time, both being steady flow fields. In fact for increasing Reynolds numbers, the Navier-Stokes Equations approach the inviscid Euler Equations, implying that the flow would converge to the inviscid solutions of Potential Flow Theory. Thus the Reynolds spectrum appears to be an ouroboros, similarly to the Riemann circle.

Lastly, we point out that in the discussion of Chaotic Advection, we have not introduced the standard mathematics in full, generally employed in the theory. This is partly because we find the existing methods to be suboptimal for actualizing the potential in its concepts, and partly because the theory in its current form is often intuitive and in need of a rigorous mathematical language. So we feel that the theory needs a complete reformulation unifying the Aref and Ott directions in the spirit of Wiggins, which is one of our goals in this paper.

So to project our intentions, the model to be presented shall employ the methods of Optimal Transport Theory to reformulate in a new language the theory of Chaotic Advection, in the setting of Potential Flows, with the purpose of creating an exact geometrical model for the evolution towards the final steady equilibrium state of Fully Developed Turbulence.

3 Fractal Potential Flows

We have introduced above the theoretical elements that we intend to weave together in this section and the next, in our effort to create an exact model for Fully Developed Turbulence and the evolution towards it. We will introduce Fractal Potential Flows as our exact model, which arise due to the intermittent transfer interaction of eddies. We define a weighted transfer operator over a certain class of stream functions, and show the existence of its unique fixed point - the invariant flow - characterized by a stream function whose rotated gradient field will serve as our model. The set of sink singularities of this flow field is shown to be an IFS fractal, explicitly resolving Mandelbrot's Conjecture.

3.1 Eddy Invariance

In Section 2.2 we have defined the stream function of an eddy with sink strength $q = C \ln \lambda$ and vortex strength or circulation $\Gamma = C\vartheta$ as

$$\psi(z) = -\frac{C \ln \lambda}{2\pi} \text{Arg}(z - p) + \frac{C\vartheta}{2\pi} \ln |z - p| \mod -C \ln \lambda$$

where $\varphi \in \mathbb{C}$, $\lambda = |\varphi| < 1$, $\vartheta = \text{Arg } \varphi$ and $C > 0$ is an arbitrary parameter. The associated velocity field (continuously extended to the entire plane) becomes

$$v[\psi](z) := \frac{C \text{Log } \varphi}{2\pi} \frac{1}{|z - p|} \frac{z - p}{|z - p|}$$

We observe that the eddy stream function is invariant under the contraction $T(z) = p + \varphi(z - p)$, meaning $T^*\psi = \psi$. This is expected geometrically, since the orbit $T^t(z_0) = p + \varphi^t(z_0 - p)$, $t \in \mathbb{R}$ traces out a logarithmic spiral centered at $p \in \mathbb{C}$, and the streamlines of constant ψ are spirals of the same pitch. So in other words, ψ is the fixed point of the pushforward transfer operator T^* . This raises the question whether ψ is unique, and if not, then what space would guarantee its uniqueness. Defining $\psi_0(z) := \text{Arg}(z - p) \mod \vartheta$ we see that $T^*\psi_0 = \psi_0$ also holds, therefore ψ cannot be unique in such a general setting. So our ultimate goal becomes finding the proper function space where uniqueness can be guaranteed, with a reasonable physical interpretation. We begin our search by deriving some fundamental properties of transfer, which will have profound implications.

Theorem 3.1 *For a similarity contraction of the form $T(z) = p + \varphi(z - p)$, $|\varphi| < 1$ the following properties hold (with differentiation in the \mathbb{R}^2 sense, and $v[\psi]$ being the continuous extension of $i\nabla\psi$).*

$$\Delta T^* = \frac{1}{\lambda^2} T^* \Delta \quad \text{and} \quad \widetilde{T^*\psi} \equiv T^*\tilde{\psi} \quad \text{and} \quad v[T^*\psi] = \frac{\varphi}{|\varphi|^2} T^*v[\psi]$$

Proof We show the above in the \mathbb{R}^2 sense first, which translates to the complex sense, using $T(z) = p + \lambda R(z - p)$, $p, z \in \mathbb{R}^2$ where R is the rotation matrix corresponding to $e^{i\vartheta}$.

We first show the third property, keeping in mind that $v[\psi](z) = i\nabla\psi(z)$ a.e. Gradient is a column vector in \mathbb{R}^2 , which corresponds to the complex vector $\partial_1 + i\partial_2$. Differentiation D results in a row vector however, so we must take a transpose, meaning $\nabla = D^T$. Applying the generalized chain rule, we get

$$\nabla(\psi \circ T^{-1}) = D(\psi \circ T^{-1})^T = \left((D\psi \circ T^{-1}) \cdot \frac{1}{\lambda} R^{-1} \right)^T = \frac{1}{\lambda} R \cdot \nabla\psi \circ T^{-1}$$

Multiplying by a 90° rotation (or i), we get the third property. To get the first property, we observe that $\Delta = \text{tr } D\nabla$, so by the chain rule, the properties of trace, and the above

$$\Delta T^*\psi = \text{tr } D\nabla T^*\psi = \text{tr } D \left(\frac{1}{\lambda} R \cdot T^*\nabla\psi \right) = \text{tr } \left(\frac{1}{\lambda} R \cdot T^* D\nabla\psi \cdot \frac{1}{\lambda} R^{-1} \right) = \frac{1}{\lambda^2} T^* \Delta\psi$$

The second property also follows from the third, observing that in the above complex sense ψ and $\tilde{\psi}$ are conjugates iff $\nabla\psi = i\nabla\tilde{\psi}$ by Section 2.2. So taking T^* of both sides, multiplying by i and dividing by λ , we get that $\nabla(T^*\psi) = i\nabla(T^*\tilde{\psi})$ implying that $\nabla(T^*\psi) = \nabla(\widetilde{T^*\psi})$ and thus $\widetilde{T^*\psi} \equiv T^*\tilde{\psi}$. \square

Among many things, the first property also implies that if ψ is a.e. harmonic then so is $T^*\psi$. Together with the second property, this implies that the space of potential flows is closed under pushforward transfer by a similarity contraction.

We end our discussion of a single eddy by emphasizing the underlying idea that one application of T^* to a flow field represented by the stream function, corresponds to one intermittent energy burst fuelling an eddy to transform the flow field along the logarithmic spiral orbits of T . If T^* can be shown to be contractive over stream functions in some complete space, then the final equilibrium fixed point stream can also be shown.

3.2 The Transfer Operator

Considering that potential flows form a vector space as discussed in Section 2.2 and by Theorem 3.1, the weighted transfer operator

$$T\psi = w_1 T_1^* \psi + \dots + w_n T_n^* \psi$$

for any IFS $\{T_1, \dots, T_n\}$, $T_k(z) = p_k + \varphi_k(z - p_k)$ and weights $w_k \in (0, 1)$, $\sum_k w_k = 1$, maps the stream function of a potential flow to another such flow. One application of T to a flow field corresponds to the random application of some $T_k^* \in \{T_1^*, \dots, T_n^*\}$ with probability $P(T\psi(z_0) = T_k^*\psi(z_0)) = w_k$ at some point z_0 in the plane. Indeed, the iteration of the transfer operator T corresponds to the intermittent randomly alternating contractive action of a system of n eddies along the orbits of T_k . Here the flow field is piecewise steady in between the energy bursts fuelling some particular eddy at a certain intermittent moment. During the steady periods, the flow field is of the form $T^L\psi_0$, $L \in \mathbb{N}$ for some initial flow ψ_0 . The partitioning intermittent moments in time t_L can be spaced in any way over the timeline. This constitutes our mathematical reformulation of Chaotic Advection.

We see that after the L -th iteration we have

$$\psi_L = T^L \psi_0 = \sum_{|a|=L} w_a T_a^* \psi_0$$

where T_a and w_a were defined in Definition 2.5. Probabilistically speaking, this implies that ψ_L is the stochastic superposition of n^L eddies with transfers T_a^* , transforming the flow field at the L -th intermittent moment t_L with probabilities $P(\psi_L(z_0) = T_a^* \psi_0(z_0)) = w_a$. We also note that by Theorem 3.1 the corresponding velocity transfer operator is

$$v[T\psi] = w_1 \frac{\varphi_1}{|\varphi_1|^2} T_1^* v[\psi] + \dots + w_n \frac{\varphi_n}{|\varphi_n|^2} T_n^* v[\psi]$$

There are two possible interpretations of the iteration of the above transfer operator - probabilistic and physical. Probabilistically it means that considering the next iterate ψ_{L+1} to be a random variable, its expected value is $T\psi_L$. A related financial interpretation is that the collective action of the eddies, is a portfolio. Physically, weighted transfer can be interpreted as weighting the effect each eddy can have on the fluid at an intermittent moment.

Succinctly, we observe that iteration corresponds to intermittency, and hereby theorize that the intermittent evolution of turbulence towards the equilibrium state of Fully Developed Turbulence, corresponds to the iteration of T ad infinitum to an invariant flow $T\bar{\psi} = \bar{\psi}$. Therefore $\bar{\psi}$ will represent our model for the fully developed turbulent flow field, and showing its unique existence now becomes our primary goal, along with finding an appropriate function space where this is possible and physically most reasonable.

Theorem 3.2 *For the weighted transfer operator T the following properties hold.*

$$\Delta T = \sum_{k=1}^n \frac{w_k}{\lambda_k^2} T_k^* \Delta \quad \text{and} \quad \widetilde{T\psi} \equiv T\tilde{\psi} \quad \text{and} \quad v[T\psi] = \sum_{k=1}^n w_k \frac{\varphi_k}{|\varphi_k|^2} T_k^* v[\psi]$$

Proof The proof follows trivially from Theorem 3.1. Note that the first property implies the preservation of harmonicity a.e. or physicality upon transfer, as discussed in Section 2.2. \square

3.3 Flow Character

In constructing the proper flow space, our inspiring objective is to ensure the unique existence of an eddy as the attractor of its generating pushforward transfer map. In resolving this question, we show a correspondence between eddies and the Dirac delta function, which hints at a possible general correspondence between stream functions and the density functions of probability measures, via Poisson's Equation. Finding the ideal kind of boundary condition will prove crucial to our quest, and meanwhile the proper space of flows shall slowly reveal itself to us.

First of all, we observe that the velocity field of the eddy discussed in Section 3.1 is characterized by the complex parameter $c = \frac{C}{2\pi} \text{Log } \varphi$ which can be extracted from the stream function ψ by the operation

$$v[\psi](z) \overline{z - p} = c \frac{z - p}{|z - p|^2} \overline{z - p} = c$$

We will denote the stream function of an eddy with character $c \in \mathbb{C}$, $\text{Re } c < 0$ and centered at $p \in \mathbb{C}$, with $q_c := -2\pi \text{Re } c$, $\Gamma_c := 2\pi \text{Im } c$ as

$$\psi_{c,p}(z) := \frac{q_c}{2\pi} \text{Arg}(z - p) + \frac{\Gamma_c}{2\pi} \ln |z - p| \pmod{q_c} \quad \text{and} \quad \psi_c := \psi_{c,0}$$

Certain flows may have a similar “eddy character” when their flow field is zoomed out, even if they exhibit varying streamline behaviour locally around the origin, as defined below.

Definition 3.1 *We say that a stream function $\psi : \mathbb{C} \rightarrow \mathbb{R}$ satisfies the boundary condition at infinity with character $c \in \mathbb{C}$, if for any $p \in \mathbb{C}$ and any sequence $(z_j) \subset \mathbb{C}$, $|z_j - p| \rightarrow \infty$ we have $\exists \lim_{j \rightarrow \infty} v[\psi](z_j) \overline{z_j - p} = c$ where the limit is taken in the complex sense. We denote this property as $\psi \in BC_c^\infty$ or $\text{char}(\psi) = c$.*

We remark that character is independent of the choice of $p \in \mathbb{C}$. To see this, take

$$\lim_{j \rightarrow \infty} |v[\psi](z_j)| = \lim_{j \rightarrow \infty} \left| \frac{v[\psi](z_j) \overline{z_j - p}}{\overline{z_j - p}} \right| = \lim_{j \rightarrow \infty} \frac{|c|}{|z_j - p|} = 0$$

so $\lim_{j \rightarrow \infty} v[\psi](z_j) = 0$. Taking some other $p' \in \mathbb{C}$ we have for the above (z_j) that $|z_j - p'| \rightarrow \infty$ since $|z_j - p| \leq |z_j - p'| + |p' - p|$ and that

$$v[\psi](z_j) \overline{z_j - p'} = v[\psi](z_j) \overline{z_j - p} + v[\psi](z_j) \overline{p - p'} \rightarrow c + 0 = c \quad \text{as } j \rightarrow \infty$$

Theorem 3.3 *Character is a linear map that is invariant under weighted transfer, or convolution with a density function over the plane.*

Proof The linearity of char is trivial by definition. Let us suppose that $\text{char}(\psi) = c$. Since for some p and (z_j) , $|z_j - p| \rightarrow \infty$ sequence $z_j - p = \varphi_k(T_k^{-1}(z_j) - p_k) + (p_k - p)$ and applying Theorem 3.1 we have that

$$\begin{aligned} v[\text{T}\psi](z_j) \overline{z_j - p} &= \sum_{k=1}^n w_k \frac{\varphi_k}{|\varphi_k|^2} v[\psi](T_k^{-1}(z_j)) \overline{\varphi_k(T_k^{-1}(z_j) - p_k) + (p_k - p)} = \\ &= \sum_{k=1}^n w_k \frac{\varphi_k \bar{\varphi}_k}{|\varphi_k|^2} v[\psi](T_k^{-1}(z_j)) \overline{T_k^{-1}(z_j) - p_k} + \sum_{k=1}^n w_k \frac{\varphi_k}{|\varphi_k|^2} v[\psi](T_k^{-1}(z_j)) \overline{p_k - p} \end{aligned}$$

which approaches $\sum_k w_k c + \sum_k 0 = c$ as $j \rightarrow \infty$ meaning $\text{char}(\text{T}\psi) = c$.

Now let ϱ be a density function over \mathbb{R}^2 and (z_j) be the above sequence. We employ scalar and complex vector Riemann integrals as follows.

$$\begin{aligned}
\nabla(\psi * \varrho)(z) &= \int_{\mathbb{R}^2} \nabla \psi(z-w) \varrho(w) \, dw \\
v[\psi * \varrho](z_j) \overline{z_j - p} &= \int_{\mathbb{R}^2} i \nabla \psi(z_j - w) \overline{(z_j - w) - p} \varrho(w) \, dw + \\
+ \int_{\mathbb{R}^2} i \nabla \psi(z_j - w) \bar{w} \varrho(w) \, dw &\rightarrow \int_{\mathbb{R}^2} c \varrho(w) \, dw + 0 = c \quad \text{as } j \rightarrow \infty
\end{aligned}$$

□

In what follows, we will denote with $L^\infty(\mathbb{R}^2)$ the set of bounded functions over the plane, and with $C_{ae}^2(\mathbb{R}^2)$ the a.e. defined and twice a.e. continuously differentiable functions ψ , for which $i\nabla\psi$ can be continuously extended to the entire plane, denoted as $v[\psi]$, furthermore the set of singularities S_ψ of $v[\psi]$ is compact and nonempty.

Theorem 3.4 *For any density function ϱ over the plane and any $c \in \mathbb{C}$, $\text{Im } c \neq 0$, there exists a $\psi \in L^\infty(\mathbb{R}^2) \cap C_{ae}^2(\mathbb{R}^2) \cap BC_c^\infty$ for which $\Delta\psi = \Gamma_c \varrho$ a.e. Furthermore, this ψ is unique up to a gradient (a.e.) and $\psi = \psi_c * \varrho$.*

Proof The existence of such a ψ is guaranteed using the Green function method of convolving the fundamental solution ψ_c ($\Delta\psi_c = \Gamma_c \delta_0$ a.e.) with ϱ which will also have a character of c by the previous theorem. So we turn to the question of uniqueness (up to a gradient). Let us suppose indirectly that $\exists \psi_{1,2} \in BC_c^\infty$, $\psi_1 \not\equiv \psi_2$ such that $\Delta\psi_1 = \Gamma_c \varrho = \Delta\psi_2$ a.e. Then $\Delta\psi = 0$ a.e. with $\psi := \psi_1 - \psi_2$. By the Divergence Theorem, for some compact $S \subset \mathbb{R}^2$ with piecewise smooth boundary

$$\int_S |\nabla \psi|^2 = \int_S |\nabla \psi|^2 + \psi \Delta \psi = \int_S \text{div}(\psi \nabla \psi) = \int_{\partial S} \psi \langle \nabla \psi, u \rangle$$

Since $\psi_{1,2}$ are bounded, then so is ψ with say $|\psi| \leq M$. So for any sequence of disks $S_j := B(p, r_j)$, $0 < r_j < r_{j+1}$, $j \in \mathbb{N}$, $\lim(r_j) = \infty$ for which $S_\psi \subset S_1$, $\partial S_\psi \cap \partial S_1 = \emptyset$

$$\int_{S_j} |\nabla \psi|^2 = \left| \int_{\partial S_j} \psi \langle \nabla \psi, u \rangle \right| \leq M \int_{\partial S_j} \left| \left\langle \nabla \psi(z), \frac{z-p}{|z-p|} \right\rangle \right| dz \leq M \int_{\partial S_j} |v[\psi](z)| \, dz$$

where the integrals are in the usual \mathbb{R}^2 sense. If we can show that this last integral vanishes as $j \rightarrow \infty$, then $\int_{\mathbb{R}^2} |\nabla \psi|^2 = 0$ and so $\nabla \psi = 0$ a.e. which would contradict $\psi_1 \not\equiv \psi_2$.

Let us define $z_j := \arg\max_{z \in \partial S_j} |v[\psi](z)|$, $j \in \mathbb{N}$ which exist by the Extreme Value Theorem, considering the compactness of ∂S_j and the continuity of $v[\psi]$ on $\partial S_j \subset \mathbb{R}^2 \setminus S_\psi$. Since $z_j \in \partial S_j$, $j \in \mathbb{N}$ we have $|z_j - p| = r_j \rightarrow \infty$, $j \rightarrow \infty$ which by $\text{char}(\psi_{1,2}) = c$ implies that $v[\psi](z_j) \overline{z_j - p} \rightarrow 0$ so

$$\int_{\partial S_j} |v[\psi](z)| \, dz \leq \int_{\partial S_j} |v[\psi](z_j)| \, dz = |v[\psi](z_j) \overline{z_j - p}| \frac{1}{|z_j - p|} 2\pi r_j \rightarrow 0 \quad \text{as } j \rightarrow \infty \quad \square$$

By the above results, we may conclude among many things that the eddy $\psi_{c,p}$ uniquely corresponds via Poisson's Equation to the Dirac delta function δ_p among all bounded functions of character c , where the uniqueness is guaranteed up to a gradient.

3.4 The Flow Space and the Measure Map

Definition 3.2 Let $X \subset \mathbb{R}^2$ be a compact set such that $F = \langle T_1, \dots, T_n \rangle \subset X$ and $c \in \mathbb{C}$ some number for which $\operatorname{Re} c < 0$, $\operatorname{Im} c \neq 0$. For a function $\psi \in C_{ae}^2(\mathbb{R}^2)$ define

$$\mu[\psi](S) = \mu_c[\psi](S) := \frac{1}{\Gamma_c} \int_S \Delta \psi \quad (S \subset \mathbb{R}^2)$$

Let the set of flows with character c be defined as

$$\Psi_{X,c} := \{\psi \in C_{ae}^2(\mathbb{R}^2) \cap BC_c^\infty : \operatorname{Ran} \psi \subset [0, q_c), \mu[\psi] \in \mathcal{P}_X\}$$

We will call $\mu : \Psi_{X,c} \rightarrow \mathcal{P}_X$ the measure map, and $\mu[\psi]$ the measure induced by ψ . Recalling the metric d from Section 2.5, for any $\psi_{1,2} \in \Psi_{X,c}$ denote

$$D(\psi_1, \psi_2) = D_{X,c}(\psi_1, \psi_2) := d(\mu[\psi_1], \mu[\psi_2]) = \sup \left\{ \frac{1}{\Gamma_c} \int_X f \Delta(\psi_1 - \psi_2) : f \in \mathcal{L}_1(X) \right\}$$

As in Section 2.2, we will consider two functions to be congruent $\psi_1 \equiv \psi_2$ iff $\nabla \psi_1 = \nabla \psi_2$ a.e. over X . Let the equivalence class of ψ be denoted as $[\psi] := \{\psi_0 \in \Psi_{X,c} : \psi \equiv \psi_0\}$ and let

$$[\Psi_{X,c}] := \{[\psi] : \psi \in \Psi_{X,c}\} = \Psi_{X,c} / \operatorname{Ker}(\mu_c)$$

Further denote $D([\psi_1], [\psi_2]) = D_{X,c}([\psi_1], [\psi_2]) := D_{X,c}(\psi_1, \psi_2)$.

Then we call $([\Psi_{X,c}], D)$ the flow space over X with character c .

Theorem 3.5 $([\Psi_{X,c}], D)$ is a well-defined metric space. The measure map $\mu : [\Psi_{X,c}] \rightarrow \mathcal{P}_X$ is an isometric isomorphism, and the weighted transfer maps $T : \Psi_{X,c} \rightarrow \Psi_{X,c}$. Furthermore, the two maps commute.

Proof First of all, we observe that $\mu : [\Psi_{X,c}] \rightarrow \mathcal{P}_X$ is clearly an isometry, since by definition $d(\mu[\psi_1], \mu[\psi_2]) = D([\psi_1], [\psi_2])$. μ is bijective iff $\forall \nu \in \mathcal{P}_X \exists! [\psi] \in [\Psi_{X,c}] : \mu[\psi] = \nu$. The equality $\mu[\psi] = \nu$ here means that with the density function ϱ belonging to $\nu \in \mathcal{P}_X$

$$\mu[\psi](S) = \frac{1}{\Gamma_c} \int_S \Delta \psi = \nu(S) = \int_S \varrho \quad \forall S \subset X$$

which is equivalent to $\Delta \psi = \Gamma_c \varrho$ a.e. on X . The unique existence of such a $\psi \in \Psi_{X,c}$ for any ϱ was shown in Theorem 3.4 up to a gradient, meaning up to a congruence or equivalence class. So we have that μ is a bijective isometry, also called an isometric isomorphism.

For the well-definition of the space, we must show that D is a metric over the set $[\Psi_{X,c}]$. D is positive, symmetric, and inherits the triangle inequality from d . The only question that remains is whether $D([\psi_1], [\psi_2]) = 0$ implies $[\psi_1] = [\psi_2]$. Since $0 = D([\psi_1], [\psi_2]) = d(\mu[\psi_1], \mu[\psi_2])$ we have that $\mu[\psi_1] = \mu[\psi_2]$, and since μ has been shown to be bijective, this implies that $[\psi_1] = [\psi_2]$ or $\psi_1 \equiv \psi_2$.

We go on to showing that T and μ commute, meaning $T\mu = \mu T$. We recall from Theorem 3.2 that

$$\Delta(T\psi)(z) = \sum_{k=1}^n \frac{w_k}{\lambda_k^2} (\Delta\psi) \circ T_k^{-1}(z)$$

and also see that for any $S \subset X \subset \mathbb{R}^2$ and $k \in \{1, \dots, n\}$, by taking Riemann integrals over \mathbb{R}^2 we have that

$$\begin{aligned} \mu[T_k^*\psi](S) &= \frac{1}{\Gamma_c} \int_S \Delta(\psi \circ T_k^{-1}) = \frac{1}{\Gamma_c} \int_S \frac{1}{\lambda_k^2} (\Delta\psi)(T_k^{-1}(z)) \, dz = \\ &= \frac{1}{\Gamma_c} \int_{T_k^{-1}(S)} \Delta\psi(w) \, dw = \mu[\psi](T_k^{-1}(S)) = (T_k^*\mu[\psi])(S) \end{aligned}$$

so taking a convex combination and using the linearity of μ we have the desired property $\mu[T\psi] = T\mu[\psi]$, $\psi \in \Psi_{X,c}$ or succinctly $\mu T = T\mu$.

Lastly, we show that T maps from $\Psi_{X,c}$ to itself. T preserves $C_{ae}^2(\mathbb{R}^2)$ since $S_{T\psi} = H(S_\psi)$, and it also preserves character according to Theorem 3.3. Since T is a convex combination, it preserves that $\text{Ran}(T\psi) \subset [0, q_c]$. Furthermore, we have that $\mu[T\psi] = T\mu[\psi] \in \mathcal{P}_X$ since $\mu[\psi] \in \mathcal{P}_X$ and the measure transfer is known to map $T : \mathcal{P}_X \rightarrow \mathcal{P}_X$ by Section 2.5. \square

Physically speaking, the above space $([\Psi_{X,c}], D)$ represents the fact that the intermittent interaction of a system of n eddies, can be considered in a locally isolated manner over a compact $X \subset \mathbb{R}^2$ of reasonable scale. $[\Psi_{X,c}]$ versus $\Psi_{X,c}$ signifies that it is only the velocity field induced by a stream function that matters. Prior to the start of their interaction, the eddies ψ_{c_k, p_k} (notation of Section 3.3) superpose initially in a flow with a certain character c as follows. Letting $\psi_0 := \sum_k w_k T_k^* \psi_{c_k, p_k}$ then by Theorem 3.1

$$v[\psi_0] = \sum_{k=1}^n w_k \frac{\varphi_k}{|\varphi_k|^2} T_k^* v[\psi_{c_k, p_k}] = \sum_{k=1}^n w_k v[\psi_{c_k, p_k}]$$

so clearly $c = \text{char}(\psi_0) = \sum_k w_k c_k$. As the intermittent interaction of the eddies progresses according to the transfer operator T , their overall character will be preserved to be this c by Theorem 3.3.

3.5 The Invariant Flow

Theorem 3.6 *The transfer operator T is contractive over $([\Psi_{X,c}], D)$. For any $\psi_0 \in \Psi_{X,c}$ the iteration $\psi_{L+1} = T\psi_L \in \Psi_{X,c}$ converges to*

$$\bar{\psi} = \psi_c * \bar{\nu} = \lim_{L \rightarrow \infty} \sum_{|a|=L} w_a \psi_c(\cdot - T_a(p)) \in \Psi_{X,c}$$

for which $T\bar{\psi} \equiv \bar{\psi}$, and here $\bar{\nu}$ is the invariant measure of Theorem 2.3 and p is the primary fixed point. We will call this $\bar{\psi}$ the invariant flow (or fractal potential flow) with respect to the IFS $\{T_1, \dots, T_n\}$ and character $c \in \mathbb{C}$.

Proof The contractivity of transfer over stream functions, follows directly from that of the measure transfer in Theorem 2.3, considering that the measure map is isometric by definition. At this point one would expect to show the completeness of $([\Psi_{X,c}], D)$ in order to conclude the unique existence of a fixed flow of T by the Banach Fixed Point Theorem. This however becomes unnecessary, considering that we have an isometric isomorphism $\mu : [\Psi_{X,c}] \rightarrow \mathcal{P}_X$ between flows and measures, which commutes with T by Theorem 3.5.

By Theorem 2.3 and the remarks we have made on it, there exists a unique $\bar{\nu} \in \mathcal{P}_X : T\bar{\nu} = \bar{\nu}$. By Theorem 3.5 there exists a unique $[\bar{\psi}] \in [\Psi_{X,c}] : \mu[\bar{\psi}] = \bar{\nu}$ and

$$\mu[T\bar{\psi}] = T\mu[\bar{\psi}] = T\bar{\nu} = \bar{\nu} = \mu[\bar{\psi}] \Rightarrow T\bar{\psi} \equiv \bar{\psi}$$

since μ is bijective. Now let us suppose indirectly that $[\bar{\psi}]$ is not a unique fixed point, meaning there is another $\psi \neq \bar{\psi}$ for which $T\psi \equiv \psi$. Then

$$\mu[\psi] = \mu[T\psi] = T\mu[\psi] \Rightarrow \mu[\psi] = \bar{\nu} = \mu[\bar{\psi}] \Rightarrow \psi \equiv \bar{\psi}$$

which is a contradiction.

Now let $\psi_0 \in \Psi_{X,c}$ be any flow, and consider the recursion $\psi_{L+1} = T\psi_L \in \Psi_{X,c}$ and define $\nu_L := \mu[\psi_L]$. Then $\nu_{L+1} = \mu[T\psi_L] = T\nu_L$ so by Theorem 2.3 we have that $\exists \lim(\nu_L) = \bar{\nu}$ in (\mathcal{P}_X, d) . Since $D([\cdot], [\cdot]) = d(\mu[\cdot], \mu[\cdot])$ and μ is injective, we have that $\exists \lim([\psi_L]) = [\bar{\psi}]$. We go on to characterizing the exact form of $\bar{\psi}$. From Section 2.5 we recall that

$$\bar{\nu} = \lim_{L \rightarrow \infty} \sum_{|a|=L} w_a T_a^* \delta_p$$

From the proof of Theorem 3.4 we see that $\psi_c \in \Psi_{X,c} : \Delta\psi_c = \Gamma_c \delta_0$ so

$$\bar{\psi} = \psi_c * \bar{\nu} = \lim_{L \rightarrow \infty} \sum_{|a|=L} w_a \psi_c * (\delta_p \circ T_a^{-1}) = \lim_{L \rightarrow \infty} \sum_{|a|=L} w_a \psi_c(\cdot - T_a(p)) \quad \square$$

We remark that $\Delta\bar{\psi} = 0$ a.e. and if $\Delta\psi_0 = 0$ a.e. then $\Delta\psi_L = 0$ a.e. ($L \in \mathbb{N}$).

Furthermore, taking $\psi_0 := \psi_{c,p}$

$$T^L \psi_{c,p} = \sum_{|a|=L} w_a T_a^* \psi_{c,p} \equiv \sum_{|a|=L} w_a \psi_c(\cdot - T_a(p)) \pmod{q_c}$$

The first equality is clear, while the second equivalence requires some consideration. It can be shown by induction that $z - T_a(p) = \varphi_a(T_a^{-1}(z) - p)$. So since $\psi_{c,p} = \psi_c(\cdot - p)$ we have

$$\psi_{c,p}(T_a^{-1}(z)) = \psi_c(T_a^{-1}(z) - p) \equiv \psi_c(z - T_a(p)) - \psi_c(\varphi_a) \pmod{q_c}$$

Summing over all addresses $|a| = L$ with factors w_a we get the earlier congruence.

Examining the intriguing quantities $\psi_c(\varphi_a)$ further, using the notations of Definition 2.5, we see that

$$\sum_{|a|=L} w_a \psi_c(\varphi_a) \equiv \frac{q_c}{2\pi} \sum_{|a|=L} w_a \vartheta_a + \frac{\Gamma_c}{2\pi} \sum_{|a|=L} w_a \ln \lambda_a \pmod{q_c}$$

Choosing the weights $w_k = \lambda_k^s$ (where $s > 0$, $\sum_k \lambda_k^s = 1$ is the so-called pseudodimension of the IFS fractal $\langle T_1, \dots, T_n \rangle$ which corresponds to its Hausdorff dimension if the open set condition holds) we have as the factor of $\frac{\Gamma_c}{2\pi}$ above the following quantity

$$-\frac{1}{s} \sum_{|a|=L} \lambda_a^s \ln \lambda_a^s = -\frac{1}{s} \sum_{|a|=L} w_a \ln w_a$$

Considering the weights $w_k = \lambda_k^s$ to be probabilities, this is the Gibbs Entropy Formula for a collection of classical particles. This can be interpreted as the entropy of the set of points $F_L = \{T_a(p) : a \in \mathcal{A}, |a| = L\}$ at each level $L \in \mathbb{N}$ during the Chaos Game evolution towards the IFS fractal attractor $F = \langle T_1, \dots, T_n \rangle = \text{Cl}\{T_a(p) : a \in \mathcal{A}\}$. Remarkably $\frac{1}{s}$ plays the role of Boltzmann's constant, hinting at some potentially deeper interpretations.

3.6 The Evolution towards Invariance

As we have reasoned in Section 3.2, the intermittent evolution towards a fully developed turbulent flow field, corresponds to the iteration of the weighted / probabilistic transfer operator towards the invariant flow $\bar{\psi}$ characterized in Theorem 3.6.

One of the advantages of our model over classical Chaotic Advection formalism, is that the intermittent / iterative steps need not occur at equal time intervals, and in fact the spacing in time can be arbitrary. An interesting question is whether the fully developed state can be achieved in finite time. The answer is, it certainly can. If the time spacing of each iteration decreases say geometrically, then the total time to reach the $L = \infty$ level iteration (corresponding to $\bar{\psi}$) will be finite. This is clearly true for any convergent sequence of time spacing.

Another question which may arise when considering the previous sections, is how the global superposition of n pushforwarded stream functions according to the transfer operator T relates to the local experimental picture of the splitting of eddies. Upon some contemplation, we may realize that the global and local viewpoints are the direct consequence of the associativity of transfer $T(T^L \psi_{c,p}) = T^L(T \psi_{c,p})$. Executing an iteration of T over the entire flow field $T^L \psi_{c,p}$ corresponds to the local intermittent splitting of the initial eddy $\psi_{c,p}$ into n eddies $T \psi_{c,p}$ and therefore the splitting of all of its level L iterates.

3.7 Equilibrium Points of the Invariant Flow

3.7.1 Sink Singularities

From the representation shown for the invariant flow

$$\bar{\psi}(z) = \lim_{L \rightarrow \infty} \sum_{|a|=L} w_a \psi_c(z - T_a(p))$$

where the limit is taken in the sense of $([\Psi_{X,c}], D)$, we see that $\bar{\psi}$ is an infinite shifted superposition of the eddy ψ_c of character c , preserved by $\bar{\psi}$. Since ψ_c is centered at the origin, each weighted eddy $w_a \psi_c(z - T_a(p))$ is centered at $T_a(p)$.

Therefore the set of sink singularities of $\bar{\psi}$ is precisely the countably infinite IFS fractal $(T_1, \dots, T_n) = \{T_a(p) : a \in \mathcal{A}\}$ where p is the primary fixed point. Since as we have reasoned $\bar{\psi}$ represents a fully developed turbulent flow field, this resolves Mandelbrot's Conjecture 2.1. Note that IFS fractals mostly have a non-integer Hausdorff dimension, but in certain special cases, such as Dragon Curves, the dimension can equal two.

Since the flow concentrates on an IFS fractal, we must focus our efforts on the study of such sets, in order to uncover the characteristics of fractal potential flows. We also emphasize that by Theorems 3.1 and 3.2, having similarity contractions is necessary for preserving the physicality of flows under transfer.

3.7.2 Saddle Points

When visualizing $T^L \psi_{c,p}$ for large enough $L \in \mathbb{N}$, one observes a thinning of the basins of attraction of each eddy center $F_L = \{T_a(p) : a \in \mathcal{A}, |a| = L\}$ which raises the question whether the thinning continues on to a width of zero. The basins are partitioned by directrices (manifolds) of the saddle points (hyperbolic equilibria) of the flow field. In the neighbourhood of each partitioning infinite separatrix (stable manifold) belonging to these saddles (the unstable one ending in sinks), the flow behaviour becomes chaotic - meaning a tracer particle starting down the streamline near one side of the separatrix, may end up at a distant sink singularity, relative to if it had started near the other side. When the flow field is considered as the phase portrait of a Hamiltonian system, then this signifies sensitivity to initial conditions near the particular separatrix. For the entire flow field to be considered chaotic, we must show that such partitioning separatrices "cover" the flow field, in some sense. We also intuit that this covering would be implied if the saddle points could be shown to be dense in the countably infinite IFS fractal (T_1, \dots, T_n) .

Conjecture 3.1 *The infinite separatrices of the saddle points of $\bar{\psi}$ are dense in the plane.*

Conjecture 3.2 *The saddle points of $\bar{\psi}$ are dense in (T_1, \dots, T_n) .*

We further conjecture that the second conjecture implies the first, which in turn guarantees the chaotic nature of the invariant flow field.

4 Visualization

When considering the transfer operator over stream functions, one encounters a proliferation of branch cuts, as the transfer iteration progresses. This can make the visualization of streamlines - the level curves of the stream function - quite difficult. Note that stream functions in $[\Psi_{X,c}]$ are only required to be almost everywhere smooth, and the transfer iteration results in a countably infinite Hutchinson union of branch cuts in $\bar{\psi}$, which pose no issue theoretically, despite the visual and algorithmic mess they create in practice. This is illustrated by Figure 2 with transfer parameters

$$p_1 = 0, p_2 = 1, \varphi_1 = 0.65e^{-\frac{2\pi}{6}i}, \varphi_2 = 0.65e^{\frac{2\pi}{4}i}, w_{1,2} = 0.5$$

Therefore in order to iteratively visualize the flow field, one seems to have no option but to resort to an iteration over the velocity field. ψ_0 can be chosen arbitrarily, as apparent from Theorem 3.6, so we choose $\psi_0 := \sum_k w_k T_k^* \psi_{c_k, p_k}$ as discussed at the end of Section 3.4, where $c_k = \text{Log } \varphi_k$ (choosing $C = 2\pi$ in Section 3.1) for an IFS of Section 2.4. The evolution in the velocity field progresses according to Theorem 3.2. So we have the following recursive iteration for the velocity field, over the flow space $([\Psi_{X,c}], D)$ with preserved character $c = \sum_k w_k \text{Log } \varphi_k$ and $X \supset \langle T_1, \dots, T_n \rangle$.

$$v_0(z) = v[\psi_0] = \sum_{k=1}^n w_k (\text{Log } \varphi_k) \frac{z - p_k}{|z - p_k|^2}$$

$$v_{L+1}(z) = v[T\psi_L](z) = \sum_{k=1}^n w_k \frac{\varphi_k}{|\varphi_k|^2} v_L(T_k^{-1}(z))$$

Once we have arrived at a large enough iteration level $L \in \mathbb{N}$, we may execute a streamline solver algorithm, resulting in a preferable image as in Figure 3. The transfer iteration is visualized in Figure 4. Note the apparent convergence to an attractor flow field, as predicted by Section 3.

In order to visualize the level curves of the corresponding potential function - the equipotential lines - we first note the correspondence in character $\text{char}(\tilde{\psi}) = -i \text{char}(\psi)$ by Definition 3.1. If one does not wish to deal with the arising sources in the conjugate to the above ψ_0 (as we did in Figure 5), we may choose simply instead $\psi_0 := \psi_{c,p}$ and thus $\tilde{\psi}_0 = \psi_{-ic,p}$ with the primary fixed point p and the above c , as discussed in Section 3.5. Then in order to arrive at the potential field at level L , we execute the same velocity recursion on $\tilde{v}_0(z) := v[\tilde{\psi}_0]$ as above on $v_0(z) = v[\psi_0]$. The iterative formula remains the same because of Theorems 3.1 and 3.2 implying the following, with $\tilde{v}_L := v[\tilde{\psi}_L]$, $L \in \{0\} \cup \mathbb{N}$.

$$\tilde{v}_0(z) = v[\tilde{\psi}_0](z) = -ic \frac{z - p}{|z - p|^2}$$

$$\tilde{v}_{L+1} = v[\tilde{\psi}_{L+1}] = v[\widetilde{T\psi_L}] = v[T\tilde{\psi}_L] = \sum_{k=1}^n w_k \frac{\varphi_k}{|\varphi_k|^2} T_k^* v[\tilde{\psi}_L] = \sum_{k=1}^n w_k \frac{\varphi_k}{|\varphi_k|^2} T_k^* \tilde{v}_L$$

Depending on the sign of $\text{Im } c$, it may be more convenient from the viewpoint of a streamline solver, to take the negative of \tilde{v}_0 so that $\text{Re char}(\tilde{v}_0) < 0$.

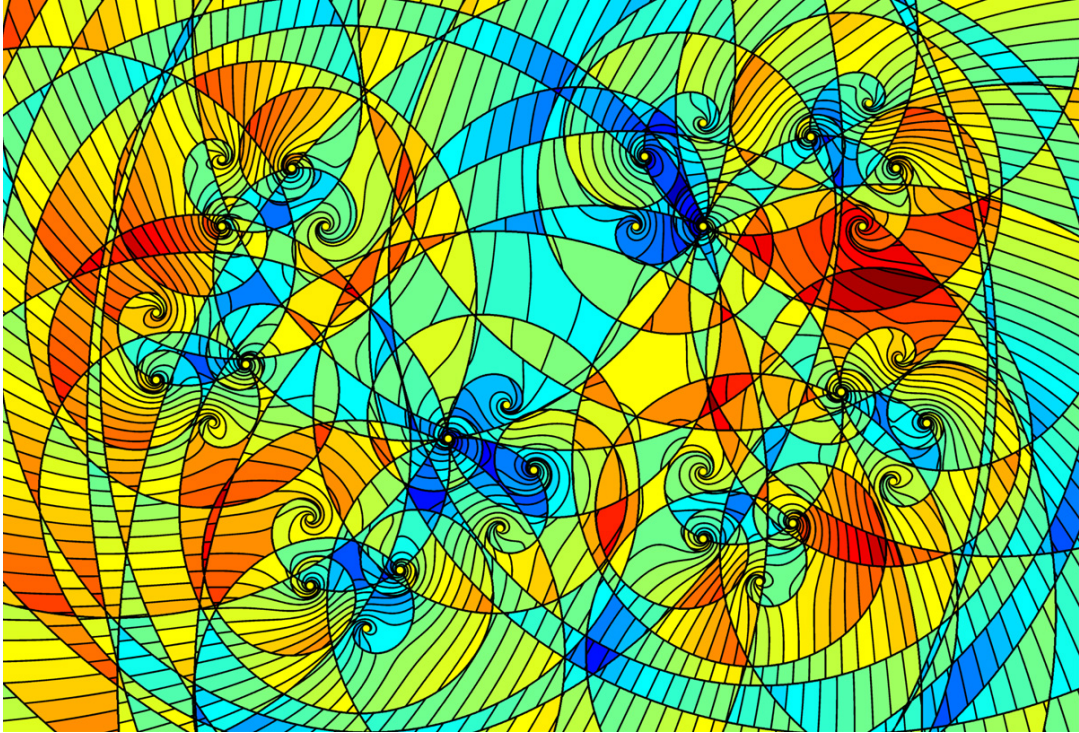


Figure 2: Level curves of the stream function $T^4\psi_0$ with logarithmic spiral branch cuts.

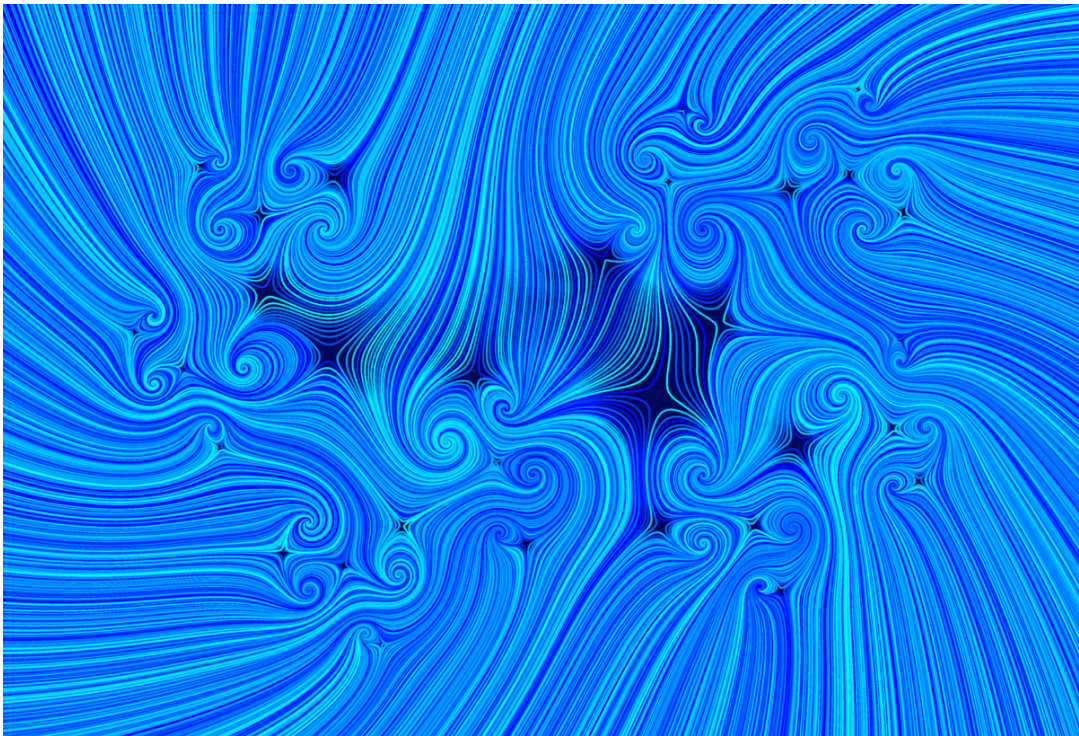


Figure 3: The same streamlines solved from the velocity field.

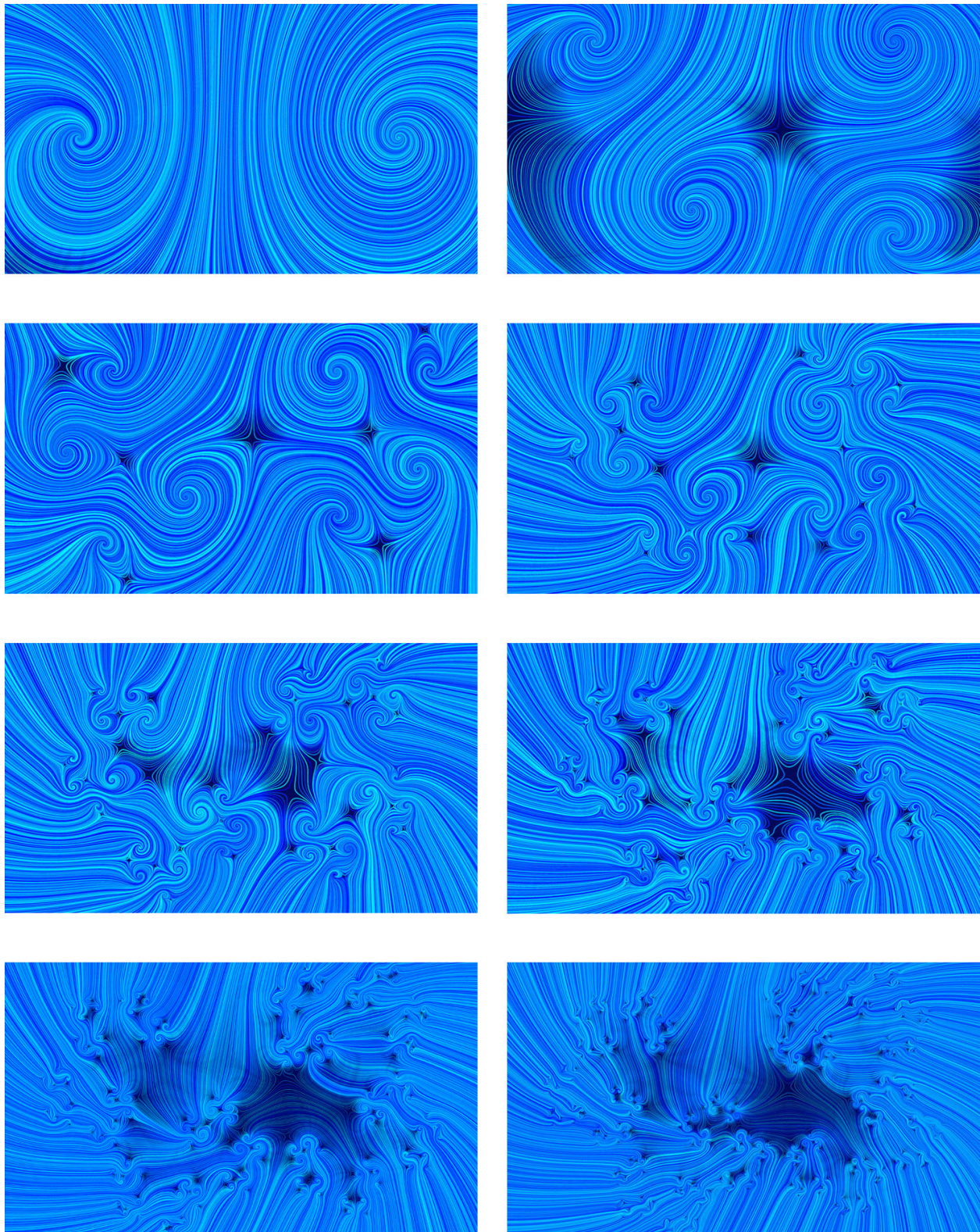


Figure 4: Converging transfer iteration from ψ_0 to $T^7\psi_0$.

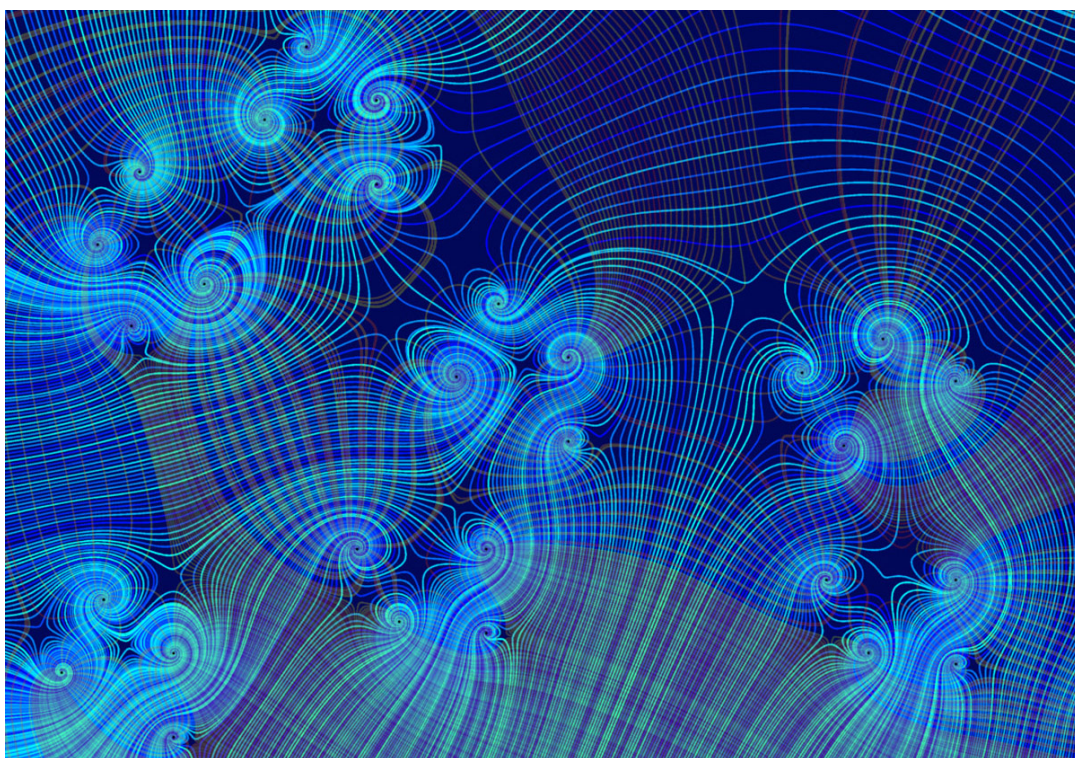
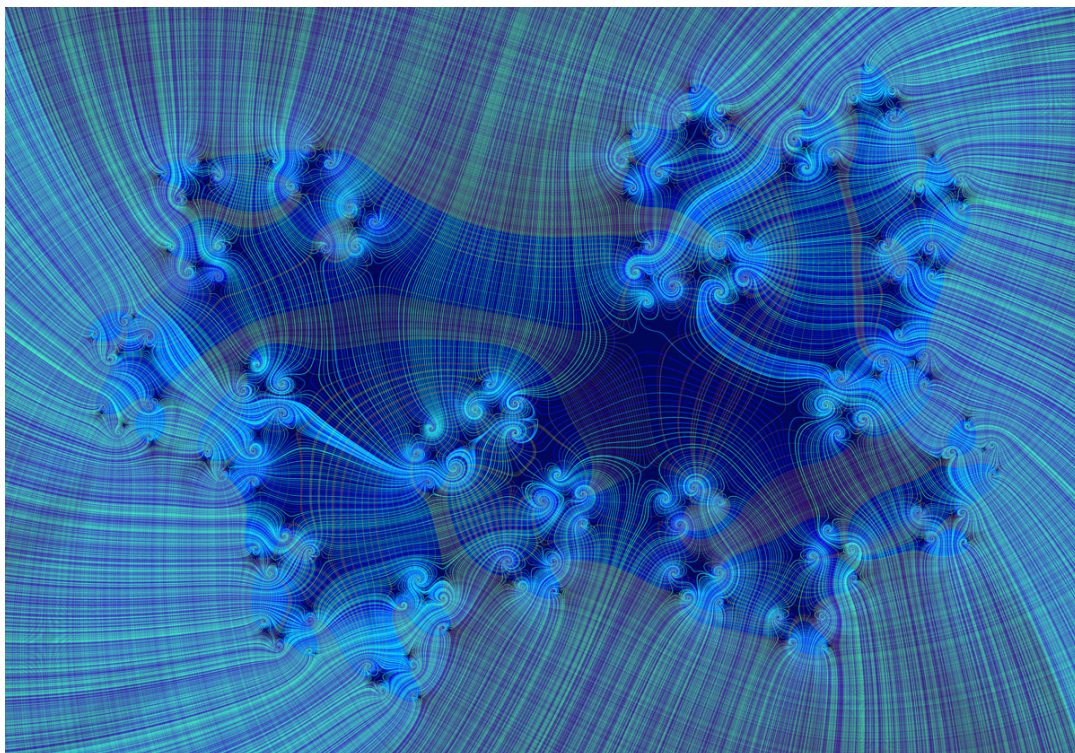


Figure 5: The equipotential lines at iteration level $L = 6$, and a detail image.

5 Conclusive Remarks

After attempting an extensive and analytical overview of Fully Developed Turbulence (FDT) in Section 2.1, we proceeded to introduce some theoretical concepts which we fused together in Section 3 in the form of Fractal Potential Flows. Our goal had been two-fold: (1) to formulate the random intermittent evolution towards FDT as the iteration of a weighted transfer operator; (2) to show that this operator has a unique attracting fixed point, modelling the FDT flow field.

As far as suggestions for future research, we mention the conjectures stated in Section 3.7.2, as well as the inspirational remarks on entropy in Section 3.5. Furthermore, the isometric isomorphism introduced in Section 3.4 between flows and probability measures, may be exploited to translate former results on measures to the language of flows, while examining the physical implications. Considering Section 3.7.1, the study of Fractal Potential Flows (and thus FDT) reduces essentially to the geometry of IFS Fractals with similarity contractions, so their investigation appears to be of fundamental importance.

We wish to express our appreciation to Dr. Sommerer and Dr. Tél for their clarifications of their work with Dr. Ott and Dr. Károlyi et al. and its relation to Aref's theory of Chaotic Advection. We express gratitude for the support of Prof. Edward R. Vrsay whose unbiased skepticism also motivated the author to chisel this attractor and its literature foundations to an arguably sufficient level of iteration, with his never sharp enough IFS maps.

We dedicate our paper to the memory of Profs. Benoit B. Mandelbrot and Hassan Aref.

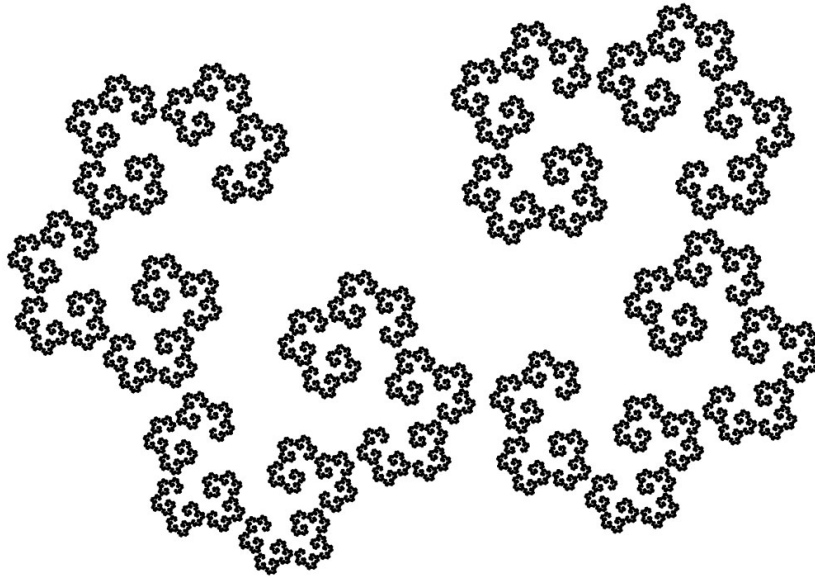


Figure 6: The IFS fractal of sink singularities ($\varphi_1 = 0.65e^{-\frac{2\pi}{6}i}$, $\varphi_2 = 0.65e^{\frac{2\pi}{4}i}$).

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